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Symmetric boundary conditions in boundary critical phenomena

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Abstract. Conformally invariant boundary conditions for minimal models on a cylinder are classified by pairs of Lie algebras (A, G) of ADE type. For each model, we consider the action of its (discrete) symmetry group on the boundary conditions. We find that the invariant ones correspond to nodes in the product graph $A \times G$ that are fixed by some automorphism. We proceed to determine the charges of the fields in the various Hilbert spaces, but, in a general minimal model, many consistent solutions occur. In the unitary models (A, A), we show that there is a unique solution with the property that the ground state in each sector of boundary conditions is invariant under the symmetry group. In contrast, a solution with this property does not exist in the unitary models of the series (A, D) and (A, E_6). A tempting interpretation of this fact is that a certain (large) number of invariant boundary conditions have unphysical (negative) classical boundary Boltzmann weights. We give a tentative characterization of the problematic boundary conditions.

1. Introduction

It has been an extremely fruitful idea to study a conformal field theory by putting it on various surfaces, with or without boundaries. Apart from the sphere, which was considered first, prime examples of non-trivial geometries include the torus [1] and the cylinder [2, 3]. They serve to probe different facets of a given conformal theory. However, the data specific to these surfaces are inextricably related to each other, and this fact provides very stringent constraints on the theory itself, allowing one, for example, to determine its field content.

For minimal conformal theories, the problem on the torus for single-valued fields has been resolved in [4]: consistent models have a periodic partition function that can be associated in a unique way with a pair (A, G) of simple Lie algebras of ADE type.

The solution of the analogous problem for the cylinder is much more recent, although early calculations in either specific models or with specific boundary conditions were carried out in [2, 3, 5]. The recent discovery in [6] of a new conformally invariant boundary condition in the three-state Potts model triggered a renewal of interest in the problem. For minimal models, its solution was given in [7, 8], and shown to be encoded in the same Dynkin graphs that specify the torus partition function.

When a model has a symmetry, necessarily discrete in this context, fields can be multiple-valued on the torus, so that non-periodic sectors exist. Furthermore, the fields transform under the symmetry group, and, upon diagonalization, can be assigned charges. All this information is encoded in frustrated partition functions, covariant under the modular group of the torus,

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a fact that can be used to, first, detect the presence of a symmetry, and then to compute the various partition functions [9, 10].

In this paper, we address the question of the action of the symmetry group on the cylinder partition functions for the minimal models. We show how the symmetry group acts on the boundary conditions, and identify the invariant (or symmetric) ones. We then study the charge assignments of the fields that occur in the presence of those boundary conditions.

Section 2 is a reminder about the minimal conformal models on a torus and on a cylinder. In section 3, we discuss the action of the symmetry group on the conformally invariant boundary conditions, which is then used in section 4 to compute frustrated partition functions on a cylinder, or equivalently the charge assignment of the boundary fields. Section 5 contains explicit formulae and computational details of a particular assignment. Its uniqueness (in fact non-uniqueness) is examined in section 6, from which we conclude that, in general, a large number of distinct charge assignments are consistent. We also derive selection rules for the boundary fusion coefficients. We finish, in section 7, with an analysis of the unitary models for which we propose an unambiguous charge assignment.

Section 7 contains the most interesting corollary of the previous sections. An analysis based on the expected consequences of the Perron–Frobenius (PF) theorem fixes a unique charge assignment in the unitary (A, A) models, which we conjecture to be the correct one. This is in sharp contrast with the models of the (A, D) and (A, E_6) series. For those, there is no consistent charge assignment that is compatible with the PF theorem, the reason being that there is no way to ensure an invariant ground state in all sectors. Motivated by the results obtained for the Potts model [6], we will interpret this phenomenon as the non-existence of positive classical Boltzmann weights for some invariant boundary conditions. A simple characterization of them suggests itself in terms of their Dynkin graph labels.

2. Minimal models

Minimal models are classified by a pair (A, G) of simply laced simple Lie algebras with coprime Coxeter numbers, p and q. One may assume that p is odd. Their periodic partition function on a torus of modulus τ is a sesquilinear form in the Virasoro characters

$$Z(A, G) = \sum_{i,j} M_{ij}\,\chi_i^*(\tau)\chi_j(\tau) \qquad M_{ij} \in \mathbb{N} \tag{2.1}$$

where i, j are labels for Virasoro highest weight representations. They lie in the Kac table $\{(r, s) : 1 \leqslant r \leqslant p-1, 1 \leqslant s \leqslant q-1\}$, in which (r, s) and $(p-r, q-s)$ must be identified. The connection with the Lie algebras is best brought out by writing the diagonal elements M_{ii} as [4]

$$Z(A, G) = \frac{1}{2} \sum_{\substack{r \in \mathrm{Exp}A \\ s \in \mathrm{Exp}G}} |\chi_{r,s}|^2 + \text{off-diagonal} \tag{2.2}$$

where r and s run over the exponents of A and G. The full expressions of the partition functions are given in [4].

The question of the symmetry group was first addressed in [9], and solved in [10] for the unitary models $|p-q| = 1$. The analysis can, however, be easily extended to the non-unitary minimal models, with the following result. With the exception of the models (A_{p-1}, A_{q-1}) with *p and q* odd, which have no symmetry at all, the other models (A, G) have a finite symmetry group Γ, which is the group of automorphisms of the Dynkin graph of G, that is, $\Gamma(G) = Z_2$ except $\Gamma(D_4) = S_3$ and $\Gamma(E_7, E_8) = \{e\}$.

When a model has a symmetry group, the fields may have a non-trivial monodromy along the two periods of the torus, transforming as $\phi(z+1) = {}^{g}\phi(z)$ and $\phi(z+\tau) = {}^{g'}\phi(z)$ for two commuting† elements $g, g' \in \Gamma$. In the Hamiltonian formalism, this amounts to give a Hilbert space $\mathcal{H}_g$ of states with a g-monodromy along the first period, which are then acted on by g' when transported along the second period. The latter action can be diagonalized, ${}^{g'}|\phi\rangle = \mathrm{e}^{2\mathrm{i}\pi Q/N}|\phi\rangle$, defining the charge Q of the field ϕ under the action of g', an element of order N.

The field content of $\mathcal{H}_g$ as well as their charges can be read off from the frustrated partition functions $Z_{g,g'}(A, G)$. These are still sesquilinear forms but with coefficients in $\mathbb{Z}(\mathrm{e}^{2\mathrm{i}\pi/|\Gamma|})$:

$$Z_{g,g'} = \mathrm{Tr}_{\mathcal{H}_g}[q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}g']. \tag{2.3}$$

Because a modular transformation mixes the two periods, it must be accompanied by a corresponding change of monodromies so that the net effect vanishes (for a fixed pair (A, G)):

$$Z_{g,g'}(\tau) = Z_{g^a g'^c, g^b g'^d}\left(\frac{a\tau+b}{c\tau+d}\right). \tag{2.4}$$

All such functions are given explicitly in [10] (with a straightforward extension to the non-unitary case). The function (2.2) corresponds to $g = g' = e$.

On a cylinder, say of length L and perimeter T, only one Virasoro algebra remains, so that the partition function is linear rather than sesquilinear in the characters [2]. Conformally invariant boundary conditions α, β must be prescribed on the two boundaries, and a monodromy condition g must be imposed along the periodic coordinate, $\phi(z+T) = {}^{g}\phi(z)$. We first consider a trivial monodromy, $g = e$.

If the time variable is defined to run along the periodic direction, the partition function is the trace of the transfer matrix $\mathrm{e}^{-TH_{\alpha,\beta}}$,

$$Z^{e}_{\alpha,\beta}(\tau) = \sum_i n^i_{\alpha,\beta}\chi_i(\tau) \qquad \tau = \mathrm{i}T/2L. \tag{2.5}$$

The integer $n^i_{\alpha,\beta}$ gives the multiplicity of the primary field with Kac label i in the Hilbert space $\mathcal{H}_{\alpha,\beta}$.

Alternatively, one may view the time evolution as going from one boundary to the other. In this case, the states on constant time surfaces belong to the bulk periodic Hilbert space $\mathcal{H}_e$, and are propagated in time from one boundary state $|\alpha\rangle$ to the other $|\beta\rangle$ (formally, also in $\mathcal{H}_e$). The partition function is then

$$Z^{e}_{\alpha,\beta}(\tau) = \langle\beta|\mathrm{e}^{-LH_e}|\alpha\rangle \tag{2.6}$$

with H_e denoting the Hamiltonian corresponding to periodic bulk sector.

The boundary states are conformally invariant, satisfying $(L_n - \bar{L}_{-n})|\alpha\rangle$ for all $n \in \mathbb{Z}$ [3]. The solutions to this equation are the Ishibashi states [11]: every highest weight representation $[i \otimes \bar{i}]$ contains exactly one such state, which we denote by $|i\rangle\rangle$, while the other representations $[i \otimes \bar{j}]$, for $i \neq j$, do not contain any. In the present situation, the Ishibashi states must be taken from the space $\mathcal{H}_e$, and hence are labelled by $\mathcal{E}_e = \{i : [i \otimes \bar{i}] \in \mathcal{H}_e\}$.

Expanding the boundary states in the basis of Ishibashi states, $|\alpha\rangle = \sum_i c^i_\alpha|i\rangle\rangle$, makes the partition function (2.6) take the form

$$Z_{\alpha,\beta}(\tau) = \sum_{i\in\mathcal{E}_e} c^i_\alpha \bar{c}^i_\beta \chi_i\left(\frac{-1}{\tau}\right). \tag{2.7}$$

† This forces us to focus on Abelian subgroups of Γ. Thus, in this paper, we consider Z_2 and Z_3 (sub)groups only.

The arguments of the characters in (2.5) and (2.7) are related by the modular transformation $\tau \mapsto \frac{-1}{\tau}$, under which the characters transform linearly through a unitary matrix S. Comparing the two formulae then yields Cardy's equation [3]

$$n^i_{\alpha,\beta} = \sum_{j\in\mathcal{E}_e} S_{i,j}\, c^j_\alpha \bar{c}^j_\beta. \tag{2.8}$$

Relations (2.8) are overdetermined for the vectors c^j, and provide a means to classify the boundary conditions $|\alpha\rangle$, to compute the spectra of $\mathcal{H}_{\alpha,\beta}$, and in turn the surface scaling dimensions. Such calculations were carried out in [2, 5, 6], but the general answer appeared only very recently in [7, 8]. Let 1 be the label corresponding to the vacuum representation, namely to $(r, s) = (1, 1) = (p-1, q-1)$.

In [8], it was observed that, upon setting $c^i_\alpha = \psi^i_\alpha/\sqrt{S_{1,i}}$ for a set of complete and orthonornal vectors ψ^i, Cardy's equation appears as an explicit diagonalization

$$n^i_{\alpha,\beta} = \sum_{j\in\mathcal{E}_e} \psi^j_\alpha \frac{S_{i,j}}{S_{1,j}} \bar{\psi}^j_\beta. \tag{2.9}$$

The matrices n^i have eigenvalues $S_{i,j}/S_{1,j}$, and a common eigenbasis is given by the vectors ψ^j. As a result, they satisfy the fusion rules

$$n^i n^j = \sum_k N^k_{ij} n^k. \tag{2.10}$$

Reversing the argument, the authors of [8] conclude that an $\mathbb{N}$-valued representation of the fusion algebra of dimension $|\mathcal{E}_e|$ provides a solution to Cardy's equation with $|\mathcal{E}_e|$ different boundary conditions. When $c^i_\alpha = \psi^i_\alpha/\sqrt{S_{1,i}}$ is an invertible matrix, this solution yields the maximal set of conformally invariant boundary conditions. Note that the boundary states $|\alpha\rangle$ are determined up to a phase, but the fact that the entries of n^i are to be positive integers leaves only a global, unobservable, phase.

For minimal models, this was all made explicit in [7]. For the model (A, G), it was shown that each node in the product Dynkin diagram $A \times G$, quotiented by an appropriate Z_2 automorphism, defines a boundary condition and vice versa. Indeed, from (2.2), the number of Ishibashi states in the periodic sector is $|\mathcal{E}_e| = \frac{1}{2}|\mathrm{Exp}\, A \times \mathrm{Exp}\, G|$, so that only half the nodes can define distinct boundary conditions. We will use the variables α, β and (a_i, b_i) as labels for the nodes of $A \times G$. The letters A and G will denote, at the same time, the Lie algebras, the Dynkin diagrams or the corresponding adjacency matrices.

As a result of the quotient of the product graph, the matrices n^i, for $i = (r, s)$, are given by [7]

$$\begin{aligned} n^i_{(a_1,b_1),(a_2,b_2)} &= (\hat{N}_r)_{a_1,a_2}(V_s)_{b_1,b_2} + (\hat{N}_r)_{a_1,a_2^*}(V_s)_{b_1,b_2^*} \\ &= n^i_{(a_1^*,b_1^*),(a_2,b_2)} = n^i_{(a_1,b_1),(a_2^*,b_2^*)}. \end{aligned} \tag{2.11}$$

In this formula, the $\hat{N}$ and the V are the fused adjacency matrices of A and G, respectively. They are defined recursively by $X_m = X_2 X_{m-1} - X_{m-2}$, with $X_1 = \mathbf{1}$ and $X_2 = A$ if $X = \hat{N}$, and $X_2 = G$ if $X = V$. Equivalently,

$$\hat{N}_r = U_{r-1}(A) \qquad V_s = U_{s-1}(G) \tag{2.12}$$

where $U_m(2\cos x) = \sin{(m+1)x}/\sin x$ is the mth Tchebychev polynomial of the second kind. The automorphism $(a, b) \mapsto (a^*, b^*)$ can be determined from the condition $n^{(r,s)} = n^{(p-r,q-s)}$ (necessary if the n^i are to satisfy the fusion algebra). It yields a^* and b^* to be given† by the non-trivial automorphism of A and G, for $G \neq D_{\mathrm{even}}, E_7, E_8$, and $b^* = b$ for $G = D_{\mathrm{even}}, E_7, E_8$.

† The automorphism * in G thus coincides with the charge conjugation in the corresponding affine algebra $\hat{G}$.

Viewing the tensor products $F^i(A, G) = \hat{N}_r \otimes V_s$ as the fused adjacency matrices of $A \times G$, the above result may be summarized by saying that n^i is a folded fused adjacency matrix of $A \times G$:

$$n^i_{\alpha,\beta} = F^i_{\alpha,\beta}(A, G) + F^i_{\alpha,\beta^*}(A, G). \tag{2.13}$$

The eigendata for the matrices A and G ensure that the matrices in (2.11) satisfy the minimal model fusion algebra. For the (A, A) models, the a_i (resp. b_i) labels run over the same set as r (resp. s), and the matrices n^i are the fusion matrices N^i themselves [3].

3. Symmetric boundary conditions

We now proceed to the analysis of the cylinder partition functions when there is a group of symmetry Γ. From now on, we thus take q even, and $G \neq E_7, E_8$.

The boundary states are combinations of periodic Ishibashi states, on which the action of Γ is known from the torus partition functions $Z_{e,g}$. This induces an action on the boundary states which one can determine. That action must be by permutations.

For the minimal models, a boundary state corresponds to a pair of nodes of A and G,

$$|(a, b)\rangle = \sum_{i \in \mathcal{E}_e} \frac{1}{\sqrt{S_{1,i}}} \psi^i(a, b) |i\rangle\rangle \tag{3.1}$$

where the ψ^i form an eigenbasis for the concrete matrices in (2.11).

Let us denote by σ the automorphisms of the Dynkin graph of G, so that every σ has fixed points. (The automorphism of the A factor has a free action, and is used to obtain a set of representatives under the * involution, see (2.11).) Each σ has a diagonalizable action on the eigenvectors ψ^i.

The action of $g \in \Gamma$ on a periodic Ishibashi state can be read off from the diagonal terms in the frustrated partition function $Z_{e,g}(A, G)$ [10]. These can be compactly presented as follows. If g has order N, and if one writes the diagonal terms in $Z_{e,g}$ as

$$Z_{e,g} = \sum_{i \in \mathcal{E}_e} \zeta_N^{Q_g(i)} |\chi_i|^2 + \cdots \tag{3.2}$$

then, for a proper choice of the ψ^i, the phase is seen to be exactly equal to the eigenvalue of ψ^i under an order N automorphism σ:

$$\psi^i(a, \sigma(b)) = \zeta_N^{Q_g(i)} \psi^i(a, b). \tag{3.3}$$

The σ that is induced by g through the previous formula is unambiguous in the models (A, G) if G is not D_4: the only non-trivial g induces the only non-trivial σ. When the D_4 algebra is involved, exactly which σ in S_3 arises from a set of charges Q_g (univocally given by $Z_{e,g}$) depends on the eigenbasis we choose. In particular, a same set of Z_2 charges can lead to the three different (but conjugate) order two σ.

It quickly follows from (3.1) and (3.3) that an order-N group element g acts on the boundary states as an order-N automorphism σ:

$$|(a, b)\rangle \longrightarrow |^g(a, b)\rangle = |(a, \sigma(b))\rangle. \tag{3.4}$$

Therefore, for any subgroup γ of Γ, the γ-symmetric boundary conditions correspond to the nodes of $A \times G$ that are fixed by a group γ of automorphisms of G. This set of nodes form a graph which we call the fixed-point graph and denote by $A \times G^\gamma$.

In particular, the boundary conditions that are invariant under a group element g correspond to the nodes in $A \times G^\sigma$, with G^σ the part of G that is fixed by the automorphism σ induced

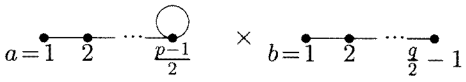

Figure 1. Fixed-point graph of an element g of order two in the $(A_{p-1}, D_{q/2+1})$ model.

by g. As before, the pairs of nodes which are related by the * automorphism define the same invariant boundary conditions. In the minimal models, the fixed-point diagrams that arise for the various choices of g are

$$\begin{array}{lll} (A_{p-1}, A_{q-1}): & T_{(p-1)/2} \times A_1 & \\ (A_{p-1}, D_{q/2+1}): & T_{(p-1)/2} \times A_{q/2-1} & (g^2 = e) \\ (A_{p-1}, D_4): & T_{(p-1)/2} \times A_1 & (g^3 = e) \\ (A_{p-1}, E_6): & T_{(p-1)/2} \times A_2 & \end{array} \tag{3.5}$$

where $T_{(p-1)/2}$ denotes the tadpole diagram obtained by quotienting A_{p-1} by its automorphism *.

For instance, the fixed-point graph of an element g of order two in the $(A_{p-1}, D_{q/2+1})$ model is graphically given by figure 1.

4. Cylinder partition functions

The consequences of a symmetry can now be pursued at the level of the partition functions. Let us suppose that α and β are two boundary conditions that are invariant under a group element g, of order N.

It implies that the transfer matrix $\mathrm{e}^{-H_{\alpha,\beta}}$ and g commute, and can be diagonalized in the same basis. The effect, on the cylinder partition function, of the insertion of g on a line connecting the two boundaries is to affect each Virasoro tower with a Nth root of unity, so that the first form (2.5) becomes

$$Z^g_{\alpha,\beta}(\tau) = \sum_i n^{(g)i}_{\alpha,\beta}\, \chi_i(\tau). \tag{4.1}$$

This shows that $n^{(g)i}$ must be related in the following way to the restriction of n^i to the g-symmetric boundary conditions: an entry of n^i equal to n becomes in $n^{(g)i}$ a sum of n Nth roots of unity.

In the second form, the boundary state $|\alpha\rangle$ is propagated to $|\beta\rangle$ by the Hamiltonian that acts on the bulk sector twisted by g, so that

$$Z^g_{\alpha,\beta}(\tau) = \langle\beta|\mathrm{e}^{-LH_g}|\alpha\rangle. \tag{4.2}$$

This formula makes it clear that the states $|\alpha\rangle$ and $|\beta\rangle$ should have a projection in the twisted Hilbert space $\mathcal{H}_g$, and being conformally invariant, must have expansions in Ishibashi states of the bulk g-sector, themselves labelled by $\mathcal{E}_g = \{i : [i \otimes \bar{i}] \in \mathcal{H}_g\}$. Setting $|\alpha\rangle = \sum_i c^{(g)i}_\alpha |i\rangle\rangle_g$, one obtains a Cardy equation

$$n^{(g)i}_{\alpha,\beta} = \sum_{j\in\mathcal{E}_g} S_{i,j}\, c^{(g)j}_\alpha \bar{c}^{(g)j}_\beta \tag{4.3}$$

for all boundary conditions which are g-symmetric.

By inspecting the torus partition functions $Z_{g,e}(A, G)$ [10] (also see the next section), one readily sees that the matrices $c^{(g)i}_\alpha$ are square, namely

$$|\mathcal{E}_g| = \tfrac{1}{2}|A \times G^\sigma| = |T \times G^\sigma| \tag{4.4}$$

where the factor $\frac{1}{2}$ accounts for the identification under *. Let us also note that, since the g-Ishibashi states form a basis for boundary states that are invariant under g, they should themselves all be neutral for consistency. This is again easily checked from $Z_{g,g}$.

The rest of this paper is devoted to a discussion of the solutions to the Cardy equation (4.3). We will suggest that the proper physical solution is a natural generalization to $g \neq e$ of the two formulae (2.9) and (2.13) for n^i.

Our first statement is that a particular solution, compatible with $n^i \equiv n^{(e)i}$, is provided, modulo a sign δ_i, by the folded fused adjacency matrices of the graph $A \times G^\sigma$:

$$\tilde{n}^{(g)i}_{\alpha,\beta} = \delta_i [F^i_{\alpha,\beta}(A, G^\sigma) + F^i_{\alpha,\beta^*}(A, G^\sigma)] \qquad \delta_i = \pm 1. \tag{4.5}$$

Here $\alpha = (a_1, b_1)$ and $\beta = (a_2, b_2)$ are pairs of nodes in $A \times G^\sigma$ (with the usual identification under *), and the automorphism * is the same as before.

When $g, \sigma \neq e$, this formula can be simplified because every b_2 in G^σ is a fixed point of *. Indeed since β is a node of $A \times G^\sigma$, b_2 is a fixed point of σ. But σ and * coincide, except for $G = D_{\text{even}}$ for which * is trivial. Thus the folding by * acts on a_2 only, resulting in an effective folding of the A factor onto a T graph (hence the graphs (3.5)). One also checks that the folded fused adjacency matrices of A_{p-1} are the fused adjacency matrices of $T_{(p-1)/2}$. Thus the matrices in (4.5) are simply proportional to the fused adjacency matrices of the fixed-point diagram

$$\tilde{n}^{(g)i}_{\alpha,\beta} = \delta_i F^i_{\alpha,\beta}(T, G^\sigma) = \delta_i U_{r-1}(T)_{a_1,a_2} U_{s-1}(G^\sigma)_{b_1,b_2}. \tag{4.6}$$

The matrices $F^i(T, G^\sigma)$ fall short of satisfying the minimal fusion algebra, but the factors δ_i can be adjusted so that the $\tilde{n}^{(g)i}$ do satisfy it.

The fusion algebra of the minimal model $\mathcal{M}(p, q)$ is polynomially generated by two generators X and Y, which one can associate with the representatives of $N^{(2,1)}$ and $N^{(1,2)}$ [12]. The other elements of the algebra are explicitly given by Tchebychev polynomials

$$N^i = U_{r-1}(X) U_{s-1}(Y) \tag{4.7}$$

and the generators must satisfy three relations:

$$U_{p-1}(X) = U_{q-1}(Y) = U_{p-2}(X) - U_{q-2}(Y) = 0. \tag{4.8}$$

The matrices $F^i(T, G^\sigma)$ have the proper form (4.7), and $T_{(p-1)/2}$ and G^σ do indeed satisfy the first two relations in (4.8). This is most easily seen by verifying that all eigenvalues satisfy the relevant equation. For instance, the eigenvalues λ_m of $T_{(p-1)/2}$ are in

$$\operatorname{spec}(T_{\frac{p-1}{2}}) = \left\{ 2\cos\frac{\pi m}{p} : 1 \leqslant m \text{ odd } \leqslant p-1 \right\} \tag{4.9}$$

and clearly satisfy $U_{p-1}(\lambda_m) = 0$.

In the same way, one computes that

$$U_{p-2}(T_{\frac{p-1}{2}}) = \mathbf{1}. \tag{4.10}$$

The corresponding calculation for G^σ yields†, in the same four cases as in (3.5),

$$\begin{aligned}
G^\sigma &= A_1 : & U_{q-2}(G^\sigma) &= (-1)^{\frac{q}{2}+1}\mathbf{1} \\
G^\sigma &= A_{\frac{q}{2}-1} : & U_{q-2}(G^\sigma) &= -\mathbf{1} \\
G^\sigma &= A_1 : & U_{q-2}(G^\sigma) &= \mathbf{1} \\
G^\sigma &= A_2 : & U_{10}(G^\sigma) &= -\mathbf{1}
\end{aligned} \tag{4.11}$$

† The adjacency matrix of A_1 is the number zero, so that its fused adjacency matrices are $U_{s-1}(0) = (-1)^{(s-1)/2}$ for s odd, and 0 for s even.

where the last line refers to the models (A_{p-1}, E_6) for which $q = 12$. Thus, except when $G^\sigma = A_1$ and when $q = 2 \bmod 4$, the last condition in (4.8) is not fulfilled.

Owing to the parity properties of the Tchebychev polynomials, $U_m(-x) = (-1)^m U_m(x)$, one easily sees that $X = (-1)^{\frac{q}{2}+1} T_{(p-1)/2}$ in the first and third cases of (4.11), and $X = -T_{(p-1)/2}$ in the second and fourth ones, together with $Y = G^\sigma$, do satisfy all three conditions and therefore generate the correct algebra.

Correspondingly, one finds that the matrices $\tilde{n}^{(g)i} = F^i(X, Y) = \delta_i F^i(T, G^\sigma)$ with the following signs,

$$\begin{array}{lll} (A_{p-1}, A_{q-1}): & \delta_i = (-1)^{(r+1)(\frac{q}{2}+1)} & \\ (A_{p-1}, D_{\frac{q}{2}+1}): & \delta_i = (-1)^{r+1} & (g^2 = e) \\ (A_{p-1}, D_4): & \delta_i = 1 & (g^3 = e) \\ (A_{p-1}, E_6): & \delta_i = (-1)^{r+1} & \end{array} \tag{4.12}$$

obey the minimal fusion algebra. Because of the signs δ_i but also because the matrices $F^i(T, G^\sigma)$ are not positive for $\sigma \neq e$ (they are, however, of constant sign), the $\tilde{n}^{(g)i}$ provide $\mathbb{Z}$-representations† of the minimal fusion algebra.

It remains to prove our earlier assertion that the so-defined $\tilde{n}^{(g)i}$ are solutions to Cardy's equation (4.3).

Since they satisfy the fusion algebra, the $\tilde{n}^{(g)i}$ must have eigenvalues given by ratios $\frac{S_{i,j}}{S_{1,j}}$ of S matrix elements. It is not difficult to see, by looking first at the partition functions $Z_{g,e}$ to get $\mathcal{E}_g$ and then by computing the ratios explicitly, that the eigenvalues of $\tilde{n}^{(g)i}$ are precisely the above ratios for $j \in \mathcal{E}_g$ (see next section). Thus the following diagonalization formulae hold:

$$\tilde{n}^{(g)i}_{\alpha,\beta} = \sum_{j \in \mathcal{E}_g} \psi^{(g)j}_\alpha \frac{S_{i,j}}{S_{1,j}} \bar{\psi}^{(g)j}_\beta \tag{4.13}$$

where the vectors $\psi^{(g)j}$ form a common orthonormal eigenbasis (also common to all fused adjacency matrices $F^i(T, G^\sigma)$ of the fixed-point diagram). This yields the value of the coefficients in (4.3)

$$c^{(g)j}_\alpha = \frac{1}{\sqrt{S_{1,j}}} \psi^{(g)j}_\alpha. \tag{4.14}$$

To complete the proof, it is enough to show that they are compatible with the n^i, in the sense that has been explained in section 3: an entry in n^i equal to n goes over, in $\tilde{n}^{(g)i}$, to a sum of n roots of unity, and moreover, $\tilde{n}^{(g)1} = \mathbf{1}$. One may verify that this is indeed the case. We omit the proof here since, to a large extent, it is given in the next section.

The formulae (4.5) and (4.13) bear much resemblance to the corresponding ones for n^i, of which they constitute a natural extension. Like the n^i, the matrices $\tilde{n}^{(g)i}$ have a graph theoretic description derived from that of n^i through the action of g, they satisfy the minimal fusion algebra, and their eigenvalues are exactly labelled by the set $\mathcal{E}_g$ which specifies the diagonal terms of the twisted partition functions $Z_{g,e}$. In a sense, this set $\mathcal{E}_g$ can also be viewed as the set of exponents of the fixed-point graph that serves to define $\tilde{n}^{(g)i}$.

5. More explicit formulae

We give here the computational details and the proofs that were missing in the previous section.

† In the case of a Z_3 symmetry group, one might expect $\mathbb{Z}(e^{2i\pi/3})$-valued representations. This is, however, excluded by the symmetry $Z^g_{\alpha,\beta} = Z^g_{\beta,\alpha}$ (time-reversal invariance), which implies the reality of $n^{(g)i}_{\alpha,\beta}$.

We begin by recalling the formula giving the S matrix elements in the minimal model $\mathcal{M}(p,q)$, for $i=(r,s)$ and $j=(r',s')$,

$$S_{i,j}=\sqrt{\frac{8}{pq}}(-1)^{rs'+r's+1}\sin\frac{\pi qrr'}{p}\sin\frac{\pi pss'}{q}. \tag{5.1}$$

We examine, in turn, each of the three infinite series.

5.1. The series (A, A)

The models (A_{p-1},A_{q-1}), p odd and q even, have the symmetry group Z_2. The invariant boundary conditions $\alpha=(a,b)$ are controlled by the tadpole graph $T_{(p-1)/2}\times A_1$, i.e. a runs from 1 to $(p-1)/2$ and $b=q/2$.

The frustrated partition functions are [10],

$$Z_{g,e}(A,A)=\frac{1}{2}\sum_{r,s}\chi^*_{r,s}\chi_{r,q-s}=\sum_{\substack{1\leqslant r\ \text{odd}\leqslant p-1\\ 1\leqslant s\leqslant q-1}}\chi^*_{r,s}\chi_{r,q-s} \tag{5.2}$$

from which it follows that the twisted Ishibashi states $|j\rangle\rangle_g$ can be labelled by

$$\mathcal{E}_g(A,A)=\{j=(m,\tfrac{q}{2}):1\leqslant m\ \text{odd}\ \leqslant p-1\}. \tag{5.3}$$

(Which representative (r,s) or $(p-r,q-s)$ we take does not matter, since the S matrix elements are the same.)

For these values of j, an easy calculation yields

$$\frac{S_{i,j}}{S_{1,j}}=(-1)^{(r+1)(\frac{q}{2}+1)}U_{r-1}\left(-2\cos\frac{\pi qm}{p}\right)U_{s-1}(0). \tag{5.4}$$

Since q is even, the numbers which appear as arguments of U_{r-1} coincide with the set (4.9) of eigenvalues of the incidence matrix $T_{(p-1)/2}$. A simple comparison with the matrices $\tilde{n}^{(g)i}$, as computed from (4.6) and (4.12),

$$\tilde{n}^{(g)i}=(-1)^{(r+1)(\frac{q}{2}+1)}U_{r-1}(T_{\frac{p-1}{2}})U_{s-1}(0) \tag{5.5}$$

shows that the eigenvalues of $\tilde{n}^{(g)i}$ are indeed the numbers in (5.4) for $j\in\mathcal{E}_g$.

As mentioned before, the matrices n^i are the fusion matrices N^i themselves [3], equal, from (2.11), to

$$n^i_{(a_1,\frac{q}{2}),(a_2,\frac{q}{2})}=N^i_{(a_1,\frac{q}{2}),(a_2,\frac{q}{2})}=U_{r-1}(T_{\frac{p-1}{2}})_{a_1,a_2} \tag{5.6}$$

for all odd s, and identically equal to zero for s even. This then leads to

$$\tilde{n}^{(g)i}_{(a_1,\frac{q}{2}),(a_2,\frac{q}{2})}=(-1)^{(r+1)(\frac{q}{2}+1)+\frac{s-1}{2}}N^i_{(a_1,\frac{q}{2}),(a_2,\frac{q}{2})}. \tag{5.7}$$

This equation shows clearly that $\tilde{n}^{(g)i}$ is compatible with n^i in the sense explained before.

5.2. The series (A, D)

All models $(A_{p-1},D_{q/2+1})$, with two coprime integers p,q and p odd as before, also have a Z_2 symmetry. The non-trivial group element g induces the automorphism σ of $D_{q/2+1}$ which exchanges the last two nodes. Therefore, the symmetric boundary states correspond to the nodes (a,b) of the fixed-point diagram $T_{(p-1)/2}\times A_{q/2-1}$, pictured in section 3, so that a is between 1 and $(p-1)/2$, and b is between 1 and $q/2-1$.

The eigenvalues of $T_{(p-1)/2}$ have been recalled earlier, while those of $A_{q/2-1}$ are well known:

$$\mathrm{spec}(T_{\frac{p-1}{2}}) = \left\{2\cos\frac{\pi m}{p} : 1 \leqslant m \text{ odd} \leqslant p-1\right\} \tag{5.8}$$

$$\mathrm{spec}(A_{\frac{q}{2}-1}) = \left\{2\cos\frac{\pi m'}{q} : 1 \leqslant m' \text{ even} \leqslant q-1\right\}. \tag{5.9}$$

The frustrated (antiperiodic) partition function on the torus is (the double sums run over $[1, p-1] \times [1, q-1]$) [10]

$$Z_{g,e}(A, D) = \sum_{\substack{r \text{ odd} \\ s \text{ even}}} |\chi_{r,s}|^2 + \sum_{\substack{r \text{ odd} \\ s = 1+\frac{q}{2} \bmod 2}} \chi^*_{r,s}\chi_{r,q-s}. \tag{5.10}$$

Thus the Kac labels of the g-Ishibashi states $|j\rangle\rangle_g$ can be chosen in the set

$$\mathcal{E}_g(A, D) = \{j = (m, m') : 1 \leqslant m \text{ odd } \leqslant p-1, 1 \leqslant m' \text{ even } \leqslant q-1\}. \tag{5.11}$$

From this, one computes

$$\frac{S_{i,j}}{S_{1,j}} = (-1)^{r+1} U_{r-1}\left(-2\cos\frac{\pi q m}{p}\right) U_{s-1}\left(-2\cos\frac{\pi p m'}{q}\right) \tag{5.12}$$

which coincide, in view of (5.8) and (5.9), with the eigenvalues of

$$\tilde{n}^{(g)i}_{\alpha,\beta} = (-1)^{r+1} U_{r-1}(T_{\frac{p-1}{2}})_{a_1,a_2} U_{s-1}(A_{\frac{q}{2}-1})_{b_1,b_2}. \tag{5.13}$$

The numbers in the set $\{2\cos\frac{\pi p m'}{q}\}$ come by pairs of opposite sign, so that the set of ratios (5.12), for fixed i, is the same whether or not there is a minus sign in the argument of U_{s-1}. Each individual ratio, however, differs by a factor $(-1)^{s+1}$, which then leads to an alternative solution $(-1)^{s+1}\tilde{n}^{(g)i}$.

Finally, the compatibility of $\tilde{n}^{(g)i}$ with the original matrices n^i can be established. In the sector of invariant boundary conditions, the latter read

$$n^i_{\alpha,\beta} = U_{r-1}(T_{\frac{p-1}{2}})_{a_1,a_2} U_{s-1}(D_{\frac{q}{2}+1})_{b_1,b_2} \tag{5.14}$$

where b_1, b_2 are restricted to lie in $[1, q/2-1]$. It is a simple matter to note the following modular identity (same values of the indices):

$$U_{s-1}(D_{\frac{q}{2}+1}) = U_{s-1}(A_{\frac{q}{2}-1}) \bmod 2. \tag{5.15}$$

This has the immediate consequence that

$$\tilde{n}^{(g)i}_{\alpha,\beta} = n^i_{\alpha,\beta} \bmod 2 \tag{5.16}$$

which shows the required compatibility.

Note that all the entries of $\tilde{n}^{(g)i}$ are in $\{0, +1, -1\}$, and that those of n^i are in $\{0, 1, 2\}$, which implies that all doubled primary fields have opposite Z_2 charges within each pair.

When $q = 6$, i.e. for the (A_{p-1}, D_4) models, Z_3 invariant boundary conditions can be investigated. They are labelled by nodes $(a, 2)$ with a in $T_{(p-1)/2}$.

The Z_3 frustrated partition functions on the torus are [10]

$$Z_{g,e}(A, D_4) = \sum_{r \text{odd}} |\chi_{r,3}|^2 + \sum_{r \text{odd}} \chi^*_{r,3}[\chi_{r,1} + \chi_{r,5}] + \text{c.c.} \tag{5.17}$$

so that the Ishibashi states in the Z_3-twisted sector have labels $j = (m, 3)$ for m odd between 1 and $p-1$.

The matrices $\tilde{n}^{(g)i}$ in (4.6) can be compared with the restriction of n^i to the sector of invariant boundary conditions, given by $U_{r-1}(T_{(p-1)/2})_{a_1,a_2}U_{s-1}(D_4)_{2,2}$. All matrices are identically zero for s even, while for s odd

$$\begin{aligned} n^i &= U_{r-1}(T_{\frac{p-1}{2}}) = \tilde{n}^{(g)i} \qquad \text{for} \quad s = 1, 5 \\ n^i &= 2U_{r-1}(T_{\frac{p-1}{2}}) \qquad \tilde{n}^{(g)i} = -U_{r-1}(T_{\frac{p-1}{2}}) \qquad \text{for} \quad s = 3. \end{aligned} \tag{5.18}$$

As in the Z_2 case, the second line shows that the doubled fields have opposite and non-zero Z_3 charge (if $\omega \neq 1$ is a third root of unity, $\omega + \omega^2 = -1$).

5.3. *The series (A, E_6)*

The models (A_{p-1}, E_6) are similar to the (A, D) models. In particular, the formula for the matrices $\tilde{n}^{(g)i}$ is the same as for the (A, D) models (with $A_{q/2-1}$ replaced by A_2).

A unique feature of the models based on E_6, however, is that some of the fields occur tripled in some boundary conditions (in addition to some others being doubled). One finds that these are the fields (r, s) with $s = 5$ and 7, in the boundary conditions corresponding to the nodes $(a, 3)$, for a in $T_{(p-1)/2}$ (with $b = 3$ the intersection of the three branches of E_6). This follows from the fused adjacency matrices $U_4(E_6)$ and $U_6(E_6)$, which, when restricted to the nodes $b = 3, 6$ corresponding to the symmetric boundary conditions, read

$$U_4(E_6) = U_6(E_6) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.19}$$

6. Uniqueness

The boundary conditions that are invariant under a group element g correspond to boundary states which have expansions in g-Ishibashi states†

$$|\alpha\rangle = \sum_{i\in\mathcal{E}_g} c_\alpha^{(g)i} |i\rangle\rangle_g. \tag{6.1}$$

The coefficients in (4.14) provide a specific solution $\tilde{n}^{(g)i}$ to Cardy's equation (4.3). As for the n^i, one may raise the question of the uniqueness of this solution.

For every g, the symmetric boundary conditions exhaust the g-Ishibashi states. It means that every other symmetric boundary state must be a linear combination of those we already have, and so must be one of them. However, since the boundary states $|\alpha\rangle$ enter Cardy's formula through scalar products, it is the boundary rays more than the boundary states which matter. Thus, the basic question is whether one retains a sensible solution if one multiplies the boundary states by phases.

Clearly, if the symmetric boundary states are multiplied by phases, $|\alpha\rangle \to \varphi_\alpha|\alpha\rangle$, the matrices change according to $\tilde{n}_{\alpha,\beta}^{(g)i} \to \varphi_\alpha\varphi_\beta^*\tilde{n}_{\alpha,\beta}^{(g)i}$, which satisfy the minimal fusion algebra for any choice of phases.

Whereas for $g = e$, the positivity of $n^{(e)i} = n^i$ forces all the phases to be equal, this is no longer the case when $g \neq e$. Since the matrices $n^{(g)i}$ are $\mathbb{Z}$ valued, the only condition one has is that the phases must be equal up to signs, $\varphi_\alpha = \epsilon_\alpha\varphi$.

For a Z_2 symmetry (or subgroup), the new matrices $\epsilon_\alpha\epsilon_\beta\tilde{n}_{\alpha,\beta}^{(g)i}$ are also solutions of the Cardy equation, because they too are compatible with the n^i. Indeed, the compatibility amounts to checking that n^i and $\tilde{n}^{(g)i}$ coincide modulo 2, which obviously remains true if signs are inserted.

† The full expansion of $|\alpha\rangle$ involves Ishibashi states from the g-twisted bulk sectors for all g which leave α invariant.

Moreover, the identity occurs in the diagonal boundary conditions only, $\alpha = \beta$, for which the signs cancel out.

In contrast, in the case of a Z_3 symmetry, the insertion of signs ϵ_α does not yield sensible solutions (as far as the minimal models are concerned). The reason is that some of the fields occur with multiplicity two. Since the corresponding entries in $n^{(g)i}$ must be real combinations of two third roots of unity, they can only be 2 or -1. Therefore, changing their sign by inserting some ϵ_α is not consistent.

Thus when the symmetry group is Z_2, there is a vast number of seemingly acceptable solutions. These various solutions differ by the charges which are assigned to the primary fields in mixed boundary conditions ($\alpha \neq \beta$). The freedom we have in choosing the ϵ_α reflects the fact that the charge normalization in mixed boundary conditions cannot be fixed *a priori*, unlike what happens for diagonal boundary conditions, in which an identity occurs.

One may try to derive more constraints on the charge normalizations by requiring that the boundary charge assignments be compatible with: (i) the charge assignments in the bulk, and (ii) the boundary field operator product coefficients.

The first requirement is a condition on the way bulk fields close to a boundary (taken to be $y = 0$) expand in boundary fields [13, 14]

$$\phi_j(x + iy) \sim \sum_{\mathrm{b.c.}\alpha} \sum_k {}^{(\alpha)}B_j^k (2y)^{h_k - 2h_j} \phi_k^{\alpha\alpha}(x) \tag{6.2}$$

where the summation on α is over all boundary conditions, not just the invariant ones. The Z_2 symmetry implies selection rules on the coefficients since a bulk field of a given parity should expand in a combination of boundary fields that transforms the same way. It means that the parity of the field $\phi_k^{\alpha\alpha}$ must match that of ϕ_j for all invariant boundary conditions α, such that ${}^{(\alpha)}B_j^k \neq 0$.

Since these expansions involve fields in diagonal boundary conditions only, the selection rules that follow are the same no matter what the signs of ϵ_α are. This does not prove, however, that the selection rules are indeed satisfied. For the diagonal models (A, A), the coefficients ${}^{(\alpha)}B_j^k$ are known explicitly [15], and it would be interesting to check directly that their values are consistent with the boundary charge assignment found here.

The second check concerns the operator algebra of the boundary fields themselves [13, 14]

$$\phi_i^{\alpha\beta}(x)\phi_j^{\beta\gamma}(x') \sim \sum_k C_{ij}^{(\alpha\beta\gamma)k} (x - x')^{h_k - h_i - h_j} \phi_k^{\alpha\gamma}(x'). \tag{6.3}$$

Restricting oneself to invariant boundary conditions α, β, γ, the discrete symmetry again implies selection rules which require that the charges given by the matrices $n^{(g)i}$ provide a grading of the boundary fusion algebra†:

$$C_{ij}^{(\alpha\beta\gamma)k} \neq 0 \quad \Longrightarrow \quad n_{\alpha,\beta}^{(g)i} n_{\beta,\gamma}^{(g)j} = n_{\alpha,\gamma}^{(g)k}. \tag{6.4}$$

It is obvious that if the matrix coefficients $\tilde{n}_{\alpha,\beta}^{(g)i}$ satisfy (6.4), the same will be true of $\epsilon_\alpha \epsilon_\beta \tilde{n}_{\alpha,\beta}^{(g)i}$, so that here too, these matrices are all consistent with the boundary operator product expansion (6.3), or else none of them is. As the discrete symmetry is expected to occur, one can be confident in the fact that the selection rules will be satisfied. Below, we give examples of selection rules in the most explicit case, namely the diagonal models. We have not shown, in general, that they are indeed satisfied, and as before, a proof which is not based on symmetry arguments would be valuable.

† We leave aside the cases where some matrix elements $n_{\alpha,\beta}^{(g)i}$ are zero without having the corresponding elements in n^i equal to zero. This happens when primary fields come in pairs of opposite charge.

In the diagonal models (A, A), the boundary conditions are in one-to-one correspondence with the chiral primary fields through their labelling by two nodes (a, b) taken in A_{p-1} and A_{q-1}. As before, the boundary conditions $(a^*, b^*) = (p-a, q-b)$ and (a, b) are to be identified. Without loss of generality, one may thus assume that the first label (the 'r-label') is odd.

The boundary operator product coefficients are known explicitly from [15], where they were proved to be equal to coefficients of the crossing matrices (in a suitable normalization)

$$C_{ij}^{(\alpha\beta\gamma)k} = F_{\beta,k}\begin{bmatrix} \alpha & \gamma \\ i & j \end{bmatrix}. \tag{6.5}$$

Since, for instance, an odd boundary field $\phi_i^{\alpha\alpha}$ cannot occur in its fusion with itself, the corresponding crossing coefficient must vanish. The verification that it does is non-trivial only when the chiral field i indeed occurs in its own bulk fusion (namely, $N_{ii}^i \neq 0$), when the primary field i indeed occurs in the diagonal boundary conditions ($n_{\alpha,\alpha}^i \neq 0$ for α invariant under Z_2), and when it is an odd field ($\tilde{n}_{\alpha,\alpha}^{(g)i} = -1$). All three conditions can be easily worked out, and yield

$$F_{\alpha,i}\begin{bmatrix} \alpha & \alpha \\ i & i \end{bmatrix} = 0 \tag{6.6}$$

for all $i = (r, s)$ such that r, s are odd, $s = 3 \bmod 4$, $r \leqslant (2p-1)/3$, $s \leqslant (2q-1)/3$, and for all $\alpha = (a, q/2)$ such that $(r+1)/2 \leqslant a \leqslant p/2$.

The simplest example where such constraints arise is the tetracritical Ising model (A_4, A_5), in which (6.6) implies (in terms of conformal weights)

$$F_{\frac{1}{15},\frac{1}{15}}\begin{bmatrix} \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} \end{bmatrix} = F_{\frac{1}{15},\frac{2}{3}}\begin{bmatrix} \frac{1}{15} & \frac{1}{15} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix} = F_{\frac{2}{3},\frac{2}{3}}\begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix} = 0. \tag{6.7}$$

More conditions can be derived in a generic diagonal model.

To summarize, the matrices $\tilde{n}^{(g)i}$ displayed in (4.6) and (4.12) yield but a particular solution to Cardy's equation. For a Z_3 symmetry, they form the only consistent solution,

$$n_{\alpha,\beta}^{(g)i} = \tilde{n}_{\alpha,\beta}^{(g)i} \qquad (g^3 = e) \tag{6.8}$$

whereas, in the case of a Z_2 symmetry, there are many more given by

$$n_{\alpha,\beta}^{(g)i} = \epsilon_\alpha\epsilon_\beta\tilde{n}_{\alpha,\beta}^{(g)i} \qquad \epsilon_\alpha = \pm 1 \qquad (g^2 = e) \tag{6.9}$$

for arbitrary signs. The effect of these signs is to reverse (or to maintain, depending to the value of $\epsilon_\alpha\epsilon_\beta$) the parity of all the fields that occur in the sector of boundary conditions α, β.

The ambiguity in the normalization of the Z_2 charges that arises due to these signs must be resolved on physical grounds. As the interpretation of the boundary fields is lacking in the general non-unitary model, it is not clear to the author what the correct requirement should be. In this context, the specific choice $\epsilon_\alpha = +1$ for all α is a minimal and natural one, as it extends nicely the corresponding formula for $g = e$, and retains much of the graph theoretic description. It also has the distinctive feature of producing matrices $\tilde{n}^{(g)i}$ of constant sign, either totally positive or totally negative†. However, in view of what follows, this may not be the correct choice.

In a unitary model, the ground state of every sector is expected to be invariant under the symmetry group, on account of the PF theorem applied to the transfer matrix. This provides a

† There is another solution in terms of matrices of constant sign, which is obtained by substituting $-G^\sigma$ for G^σ in formula (4.6) giving $\tilde{n}^{(g)i}$. The substitution has no effect when $G^\sigma = A_1$, since the associated adjacency matrix is the number zero, while in the other cases, it causes the matrices $\tilde{n}^{(g)i}$ to be multiplied by $(-1)^{s+1}$. This sign can be seen to be in the line of the previous discussion, because it is equal to $(-1)^{s+1} = \epsilon_\alpha\epsilon_\beta$ with $\epsilon_\alpha = (-1)^{b+1}$ if $\alpha = (a, b)$. The existence of this solution is a consequence of a non-trivial automorphism of the graph G^σ.

well-defined criterion to fix the normalization of the charges, and therefore, the physical value of the signs ϵ_α. We will use this criterion as a guide, in order to see if a particular set of values ϵ_α emerges from this point of view.

7. Unitary models

In this last section, we explore the possibility of fixing the value of the signs ϵ_α by using the criterion we have just mentioned: if the continuum limit is smooth enough, it is expected that the consequences of the PF theorem on the finite-dimensional transfer matrix be maintained in the corresponding conformal field theory. In particular, for all invariant boundary conditions, the ground state of the Hamiltonian $H_{\alpha,\beta}$ (the primary field of lowest conformal dimension in $\mathcal{H}_{\alpha,\beta}$) should be non-degenerate and (hence) invariant under the symmetry group. In short, we will call this the PF criterion. As already said, it is automatically satisfied in the diagonal boundary conditions.

Thus we look for a set of ϵ_α such that the Z_2 charge assignment meet the PF criterion. Incidentally, when the symmetry group is Z_3, there is only one consistent charge assignment (see the previous section). In that case, we will merely check whether the PF criterion is satisfied.

The outcome of this investigation is somewhat surprising. The unitary diagonal models are the only ones where the PF criterion can be met, for a unique choice of ϵ_α. In all other unitary models, there is no way in which it can be fulfilled, if one insists that it be valid in all sectors. A physical interpretation of this will be proposed†. Nonetheless, for all those models but two, we will see that a unique set of ϵ_α is singled out by demanding a minimal violation of the PF criterion.

We recall that the conformal weight of a primary field labelled by $i = (r, s)$ is equal to

$$h_{r,s} = \frac{(qr - ps)^2 - (p - q)^2}{4pq}. \tag{7.1}$$

Throughout this section, we will take p odd and $q = p \pm 1$ even. Then the smallest conformal weights correspond, in ascending order, to $i = (1, 1), (2, 2), (3, 3), \ldots$.

7.1. *The unitary series (A, A)*

The only boundary primary fields that occur in the diagonal models have their s-label odd (see (5.7)). Since the identity (1, 1) does not appear in mixed boundary conditions, the primary with the lowest weight that can possibly occur in mixed boundary conditions corresponds to (3, 3), and consequently, the off-diagonal entries of

$$n^{(g)(3,3)}_{\alpha,\beta} = n^{(g)(3,3)}_{(a_1,\frac{q}{2}),(a_2,\frac{q}{2})} = -\epsilon_{a_1}\epsilon_{a_2} U_2(T_{\frac{p-1}{2}})_{a_1,a_2} \tag{7.2}$$

must be positive. The off-diagonal matrix coefficients $U_2(T)_{a_1,a_2}$ equal one if $|a_1 - a_2| = 2$ or if $\{a_1, a_2\} = \{(p-3)/2, (p-1)/2\}$, and zero otherwise (it counts the number of paths of length 2 going from a_1 to a_2 on the graph $T_{(p-1)/2}$). Thus one obtains the condition $\epsilon_{a_1}\epsilon_{a_2} = -1$ for all these pairs. This fixes the vector ϵ_a in a unique way (up to a global sign that does not matter) as

$$\epsilon_a = (\ldots, +1, +1, -1, -1, +1, +1, -1, -1, +1). \tag{7.3}$$

For these specific signs, one may then verify that in the remaining mixed boundary sectors (those for which (3,3) does not occur), the field of lowest weight indeed has a parity +1 (zero

† I am indebted to Gerard Watts for a clarifying discussion about this issue.

charge). To do that, one can first observe that any mixed boundary sector has its field of lowest weight in $\{(r, s) : 3 \leqslant r = s \text{ odd} \leqslant p-2\}$. The next point is to note that $U_{r-1}(T)_{a_1,a_2}$ is zero unless the nodes a_1, a_2 can be related by a path of length $r-1$. If the two nodes cannot be connected by a shorter path, it follows from (7.3) that $\epsilon_{a_1}\epsilon_{a_2} = (-1)^{(r-1)/2}$, so that the numbers

$$n^{(g)(r,r)}_{(a_1,\frac{q}{2}),(a_2,\frac{q}{2})} = \epsilon_{a_1}\epsilon_{a_2}(-1)^{(r-1)/2}U_{r-1}(T)_{a_1,a_2} \tag{7.4}$$

are positive (or zero). The fact that a_1 and a_2 can be connected by a shorter path means that the field (r, r) is not the primary with the lowest weight in that sector, and we are back to the first case.

Since the PF criterion can be satisfied in all sectors for a unique set of ϵ_α it is tempting to conjecture that these are the correct physical values. The charge content in the various sectors of the unitary diagonal models would then be given by

$$n^{(g)i}_{(a_1,\frac{q}{2}),(a_2,\frac{q}{2})} = \epsilon_{a_1}\epsilon_{a_2}\tilde{n}^{(g)i}_{(a_1,\frac{q}{2}),(a_2,\frac{q}{2})} \tag{7.5}$$

with the signs (7.3), and the $\tilde{n}^{(g)i}$ as in (5.7).

7.2. The unitary series (A, D)

The same calculations can be carried out for the unitary models of the (A, D) series, with, however, different results. To illustrate it most clearly, we will start with the simplest case, namely (A_4, D_4), corresponding to the critical three-Potts model ($p = 5$, $q = 6$).

A set of Z_2-symmetric boundary conditions is provided† by the so-called A, BC, Free and New [6]. They correspond, respectively, to the nodes $(1, 1)$, $(2, 1)$, $(1, 2)$ and $(2, 2)$. (Free and New, being fully invariant under S_3, must correspond to $b = 2$, which is the only node of D_4 invariant under S_3.) Together they define ten different sectors.

It is not difficult to find the field with lowest weight in each of these sectors, and then compute the parity they are assigned by the matrices $\tilde{n}^{(g)i}$ computed in section 5. Writing these in a matrix $\tilde{M}$, one obtains (indices are A, BC, Free, New)

$$\tilde{M}_{\alpha,\beta} = \left(\tilde{n}^{(g)i\,\min}_{\alpha,\beta} : \min_{i\in\mathcal{H}_{\alpha,\beta}} h_i = h_{i\,\min}\right)_{\alpha,\beta} = \begin{pmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & -1 \\ +1 & -1 & +1 & 0 \\ -1 & -1 & 0 & +1 \end{pmatrix}. \tag{7.6}$$

The zeros are due to the partition function (superscripts are the conformal weights)

$$Z_{\text{Free,New}} = 2\chi^{1/15}_{3,3} + \chi^{2/5}_{3,5} + \chi^{7/5}_{3,1} \tag{7.7}$$

which shows that the ground state in that sector is doubly degenerate, the two states having opposite parities.

The above matrix makes it clear that the charge assignment implied by $\tilde{n}^{(g)i}$ does not satisfy the PF criterion in all sectors, either because the ground state is not invariant, or because it is degenerate. One may try to find values for ϵ_α that render the non-degenerate ground states invariant, but one easily sees that it is not possible: no values for ϵ_α can be found so that $\tilde{M}_{\alpha,\beta} \geqslant 0$ for all α, β.

One can relax our demands by looking for a set of ϵ_α which minimizes the number of sectors that violate the PF criterion. One then finds that the minimal number of such sectors, which we call non-PF, is equal to

$$N_{\text{non-PF}} = 2. \tag{7.8}$$

† The model has eight conformally invariant boundary conditions which are invariant under a Z_2, but not under the same Z_2. One finds three groups of four boundary conditions that are simultaneously invariant under the same Z_2. They clearly correspond to the three conjugate Z_2 subgroups of S_3, the automorphism group of D_4.

This number is realized for $\epsilon_\alpha = (+1, -1, +1, -1) = (+1, -1)_a \otimes (1, 1)_b$, the two non-PF sectors being BC, New and Free, New. Indeed, for these ϵ_α, one obtains

$$\epsilon_\alpha \epsilon_\beta \tilde{M}_{\alpha,\beta} = \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & -1 \\ +1 & +1 & +1 & 0 \\ +1 & -1 & 0 & +1 \end{pmatrix}. \tag{7.9}$$

Let us also notice that if one excludes just one boundary condition, namely 'New', the expected consequences of the PF theorem are indeed verified. Thus in this case, the minimal number of boundary conditions that have to be excluded for this to be true is equal to 1.

Finally, one may note that $\epsilon_\alpha = (+1, -1, +1, +1)$ share the same properties, the two non-PF sectors now being A, New and Free, New.

In any case, one must conclude that the transfer matrix, in certain sectors of boundary conditions, does not satisfy the conditions of the PF theorem. There can be only two reasons for this: either the transfer matrix is not irreducible†, or else it contains negative entries, implying that some of the boundary Boltzmann weights are negative (or both).

That the first condition fails is unlikely because the periodic transfer matrix is irreducible and because the boundary conditions are undecomposable. So one should favour the second alternative, which points to the unphysical nature of some of the boundary conditions, their classical description requiring negative Boltzmann weights. We note that a boundary condition α which is described by negative Boltzmann weights does not necessarily lead to unphysical (negative, non-PF) partition functions. Whether or not this is the case depends on which other boundary condition is associated with α.

The appearance of negative classical boundary Boltzmann weights to describe the New boundary condition in the critical three-Potts model has been discussed in [6], and is confirmed by the explicit calculation of the critical boundary weights [17].

As we shall see, what is true in the three-Potts model is true in all unitary models of the (A, D) series. No values for the ϵ_α exist which make all sectors satisfy the PF criterion, but a suitable choice, unique, contrary to the above case, of ϵ_α minimizes the number of sectors which do not satisfy it. As above, we will take the point of view that these features are the signal that a certain number of boundary conditions are unphysical, because they require negative Boltzmann weights for their classical description.

We have not carried out the analysis of the whole series, but instead we have investigated the first eight models, up to $p = 13$ and $q = 12$, with the following results.

In each of these models, we have determined the minimal number $N_{\rm unphys}$ of boundary conditions that must be disregarded in order to satisfy the PF criterion in all the sectors involving the remaining ones. This uniquely singles out a set of boundary conditions, which naturally qualifies as the set of unphysical boundary conditions. This also determines unique values of the ϵ_α for the physical ones. The values of ϵ_α for the unphysical α are then fixed (uniquely, except in the three-Potts model) by requiring a minimal number of non-PF sectors (which necessarily correspond to one or two unphysical boundary conditions). That minimal number is denoted by $N_{\rm non\text{-}PF}$. The results are as follows.

In the model $(A_{p-1}, D_{q/2+1})$ (we have looked at the eight models corresponding to $6 \leqslant q \leqslant 12$), the number $N_{\rm unphys}$ only depends on the rank of the D factor. It increases rather quickly since it is equal to 1, 3, 6 and 10 for the two models involving the algebra D_4, D_5, D_6 and D_7, respectively. We found that the unphysical boundary conditions form the set

† The unicity of the largest eigenvalue is only guaranteed for non-negative primitive matrices [16]. Under mild assumptions on the transfer matrix, its irreducibility is sufficient.

(the labelling of the nodes is as in the figure of section 3)

$$\left\{\alpha = (a, b) \in T_{\frac{p-1}{2}} \times A_{\frac{q}{2}-1} : a + b \geqslant \frac{p+3}{2}\right\}. \tag{7.10}$$

Moreover, the signs which make the number of non-PF sectors minimal are unique and given by

$$\begin{aligned}\epsilon_\alpha &= (+1, -1, +1, -1, \ldots)_a \otimes (1, 1, 1, \ldots)_b \\ &= (-1)^{a+1} \qquad \alpha = (a, b).\end{aligned} \tag{7.11}$$

As pointed out above, in the model (A_4, D_4), there is another solution $\epsilon_\alpha = (+1, -1, +1, +1)$, which, however, appears to contradict the duality relations (see below).

We have determined $N_{\text{non-PF}}$ by mere counting, and found that it equals 2, 3, 11, 15, 36, 46, 89, 109 for the first eight models, ordered as (A_4, D_4), (A_6, D_4), $(A_6, D_5), \ldots$. (By symmetry, the sectors (α, β) and (β, α) are identical and count for one.)

These results strongly suggest the general pattern for the whole (A, D) series, in which the number of unphysical boundary conditions in (7.10) equals a binomial coefficient

$$N_{\text{unphys}}(A, D_{\frac{q}{2}+1}) = \binom{\frac{q}{2}-1}{2}. \tag{7.12}$$

This is a large number since essentially half the invariant boundary conditions would have to be discarded as classically unphysical. Some more numerology also shows that the number of non-PF sectors fits the simple formula

$$N_{\text{non-PF}}(A_{q\mp1-1}, D_{\frac{q}{2}+1}) = \left\{\left(\frac{q-2}{4}\right)^4\right\} + \frac{q(q-2)(q\mp2)(q-4)}{192} \tag{7.13}$$

where $\{x\}$ is the integer closest to x. The two numbers in the rhs of the previous equation have, separately, a well-defined meaning: the first one is the number of sectors where the ground state is non-degenerate but odd under the Z_2 symmetry, while the second one gives the number of sectors where the ground state is doubly degenerate.

The reader may wish to check the above assertions in a less simple instance than the Potts model. A good example is to consider the (A_6, D_5) model, for which one computes (in the tensor product basis)

$$\tilde{M}_{\alpha,\beta} = \begin{pmatrix} +1 & -1 & +1 & +1 & -1 & +1 & +1 & -1 & +1 \\ -1 & +1 & -1 & -1 & +1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 & -1 & +1 & +1 & -1 & 0 \\ -1 & +1 & -1 & -1 & +1 & +1 & -1 & 0 & -1 \\ +1 & -1 & -1 & +1 & +1 & +1 & 0 & -1 & -1 \\ +1 & -1 & +1 & +1 & -1 & 0 & +1 & 0 & 0 \\ -1 & +1 & +1 & -1 & 0 & -1 & 0 & +1 & 0 \\ +1 & +1 & +1 & 0 & -1 & -1 & 0 & 0 & +1 \end{pmatrix}. \tag{7.14}$$

The values of ϵ_α mentioned in (7.11) are nothing but the first line of $\tilde{M}_{\alpha,\beta}$, and the boundary conditions to discard label the rows and columns 6, 8 and 9, which correspond, in terms of the fixed-point graph $T_3 \times A_3$, to the pairs of nodes $(a, b) = (3, 2)$, $(2, 3)$ and $(3, 3)$, as given by (7.10). There are six zeros in the upper triangular part of $\tilde{M}_{\alpha,\beta}$, which is the value of the second summand of (7.13).

All this leads to the reasonable guess that (7.11) might give the correct physical values of the ϵ_α. Inserted in (6.9), it not only determines the parities of all primaries in the sectors where the PF criterion is satisfied, but it also points to the boundary conditions that can have a problematic lattice interpretation. These conjectural statements must be confirmed or dismissed by the explicit calculation of the boundary Boltzmann weights. The results obtained so far seem to give some support to our conjecture [18].

Assuming this conjecture, it is not difficult to give an explicit formula for the parities. From (4.6), (4.12) and (7.11), they are determined from

$$n^{(g)i}_{(a_1,b_1),(a_2,b_2)} = (-1)^{a_1+a_2+r+1} U_{r-1}(T_{\frac{(p-1)}{2}})_{a_1,a_2} U_{s-1}(A_{\frac{q}{2}-1})_{b_1,b_2}. \tag{7.15}$$

The matrices $U_{r-1}(T_{(p-1)/2})$ are all positive, unlike the $U_{s-1}(A_{q/2-1})$, which are positive for $s < q/2$, negative for $s > q/2$, and identically zero for $s = q/2$, on account of $U_{q-s-1}(A_{q/2-1}) = -U_{s-1}(A_{q/2-1})$.

Putting all these observations together, one can conclude that the paired fields have opposite Z_2 parities within each pair (as already pointed out), and that the parity of an unpaired field in the sector of boundary conditions α, β is equal to

$$g(\phi_i^{\alpha\beta}) = \begin{cases} (-1)^{a_1+a_2+r+1}\phi_i^{\alpha\beta} & \text{if} \quad s < q/2 \\ (-1)^{a_1+a_2+r}\phi_i^{\alpha\beta} & \text{if} \quad s > q/2. \end{cases} \tag{7.16}$$

In the critical three-Potts model for instance, one finds the following frustrated partition functions (in terms of the conformal weights):

$$Z^g_{\mathrm{A,A}} = \chi_0 - \chi_3 \tag{7.17}$$
$$Z^g_{\mathrm{A,BC}} = \chi_{2/5} - \chi_{7/5} \tag{7.18}$$
$$Z^g_{\mathrm{A,Free}} = \chi_{1/8} - \chi_{13/8} \tag{7.19}$$
$$Z^g_{\mathrm{BC,BC}} = \chi_0 - \chi_3 - \chi_{2/5} + \chi_{7/5} \tag{7.20}$$
$$Z^g_{\mathrm{BC,Free}} = \chi_{1/40} - \chi_{21/40} \tag{7.21}$$
$$Z^g_{\mathrm{Free,Free}} = \chi_0 - \chi_3 + \chi_{2/3} - \chi_{2/3^+} \tag{7.22}$$
$$Z^g_{\mathrm{A,New}} = \chi_{1/40} - \chi_{21/40} \tag{7.23}$$
$$Z^g_{\mathrm{New,New}} = \chi_0 - \chi_3 - \chi_{2/5} + \chi_{7/5} + \chi_{2/3} - \chi_{2/3^+} + \chi_{1/15} - \chi_{1/15^+}. \tag{7.24}$$

These functions are computed using the ϵ_α given in (7.11), and appear to be consistent with the duality of the model [6]. For instance, the equality

$$Z_{\mathrm{BC,Free}} = Z_{\mathrm{A,New}} \tag{7.25}$$

is maintained for the frustrated partition functions, while

$$Z_{\mathrm{Free,Free}} = Z_{\mathrm{A,A}} + Z_{\mathrm{A,B}} + Z_{\mathrm{A,C}} \tag{7.26}$$

becomes $Z^g_{\mathrm{Free,Free}} = Z^g_{\mathrm{A,A}}$ since $Z^g_{\mathrm{A,B}} = Z^g_{\mathrm{A,C}} = 0$.

The use of the other solution $\epsilon_\alpha = (+1, -1, +1, +1)$ has the effect of multiplying by -1 the partition functions of all sectors with one 'New', so that $Z^g_{\mathrm{A,New}}$ would be minus the expression in (7.23), contradicting the duality relation (7.25).

There is a Z_3 symmetry in two models only, namely the critical and tricritical three-Potts models (A_4, D_4) and (A_6, D_4). They possess, respectively two ('Free' and 'New') and three invariant boundary conditions, namely $\alpha = (a, 2)$ for a a node of T_2 and T_3. The relevant $\tilde{M}$ matrices are equal to

$$\tilde{M}_{\alpha,\beta} = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} +1 & +1 & -1 \\ +1 & +1 & -1 \\ -1 & -1 & +1 \end{pmatrix} \tag{7.27}$$

where a -1 sign indicates that the corresponding sector has two degenerate ground states, of opposite and non-zero charge (none of them is invariant under the Z_3).

In the first case (the (A_4, D_4) model), it is the second boundary condition $(2, 2)$ (i.e. 'New') that appears to be unphysical, while in the second case, it is the third boundary condition $(3, 2)$. This should not be surprising since they are precisely the boundary conditions which were unphysical from the Z_2 point of view: from (7.10), $\alpha = (a, 2)$ was to be discarded if $a + 2 \geqslant (p + 3)/2$, that is, if $a = (p - 1)/2$. Therefore, the boundary conditions which were causing problems for the Z_2 charges also cause problems for the Z_3 charges.

7.3. The unitary models (A, E_6)

We will content ourselves with making a few comments on the two unitary models (A_{10}, E_6) and (A_{12}, E_6) ($p = 11$ or 13, and $q = 12$).

As we have said above, the models involving the E_6 algebra have the peculiarity of possessing primary fields that occur with multiplicity 1, 2 and 3. It turns out that the same is true of the ground state in various sectors. Let us examine, in some detail, the simplest model (A_{10}, E_6).

That model possesses ten invariant boundary conditions, labelled as $\alpha = (a, b)$ with $a = 1, 2, \ldots, 5$ a node of T_5, and $b = 3, 6$ a node of the A_2 subgraph of E_6, fixed by its non-trivial automorphism. One can compute, as before, the matrix $\tilde{M}_{\alpha,\beta}$ which collects those entries of $\tilde{n}^{(g)i}_{\alpha,\beta}$ for which i is the lowest weight primary in the sector α, β. The result is

$$\tilde{M}_{\alpha,\beta} = \begin{pmatrix} +1 & 0 & 0 & +1^* & -1^* & +1 & -1 & -1 & +1 & 0 \\ 0 & +1 & +1^* & 0 & -1^* & -1 & -1 & -1 & 0 & +1 \\ 0 & +1^* & +1 & -1^* & 0 & -1 & -1 & 0 & -1 & +1 \\ +1^* & 0 & -1^* & +1 & 0 & +1 & 0 & -1 & +1 & -1 \\ -1^* & -1^* & 0 & 0 & +1 & 0 & +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 & 0 & +1 & -1 & -1 & +1 & -1 \\ -1 & -1 & -1 & 0 & +1 & -1 & +1 & +1 & -1 & -1 \\ -1 & -1 & 0 & -1 & +1 & -1 & +1 & +1 & -1 & -1 \\ +1 & 0 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 \\ 0 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 \end{pmatrix} \tag{7.28}$$

where the stars mean that the ground state in the corresponding sector is three times degenerate, the number ± 1 being the sum of the three parities. As before, a zero indicates that there are two degenerate ground states with opposite parity.

We can repeat what we did for the (A, D) series, and look for a set of ϵ_α which minimizes the violation of the PF criterion.

By varying the ϵ_α, one finds that the minimal number of non-PF sectors is equal to 21, and that the non-PF sectors have at least one boundary condition in the set

$$\{(2, 3), (3, 3), (4, 3), (5, 3), (5, 6)\} \tag{7.29}$$

in terms of the nodes of $T_5 \times A_2$ (they correspond to the rows and columns 2–5, 10). So these five boundary conditions can presumably be called unphysical in the sense of the previous section. Hence

$$N_{\text{unphys}}(A_{10}, E_6) = 5 \qquad N_{\text{non-PF}}(A_{10}, E_6) = 21. \tag{7.30}$$

There are four solutions for the ϵ_α for which these values can be realized. Among them, the most symmetrical one is $\epsilon_\alpha = (+1, -1, -1, +1, -1) \otimes (1, 1)$.

The other model (A_{12}, E_6) is similar. One finds

$$N_{\text{unphys}}(A_{12}, E_6) = 5 \qquad N_{\text{non-PF}}(A_{12}, E_6) = 27. \tag{7.31}$$

The presumably unphysical boundary conditions correspond to the nodes $(3, 3)$, $(4, 3)$, $(5, 3)$, $(6, 3)$, $(6, 6)$ of $T_6 \times A_2$. The signs for which these numbers are reached are unique and given by $\epsilon_\alpha = (+1, -1, +1, +1, -1, +1) \otimes (1, 1)$.

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References

[1] Cardy J L 1986 *Nucl. Phys.* B **270** 186
[2] Cardy J L 1986 *Nucl. Phys.* B **275** 200
[3] Cardy J L 1989 *Nucl. Phys.* B **324** 581
[4] Cappelli A, Itzykson C and Zuber J-B 1987 *Commun. Math. Phys.* **113** 1
[5] Saleur H and Bauer M 1989 *Nucl. Phys.* B **320** 591
[6] Affleck I, Oshikawa M and Saleur H 1998 *J. Phys. A: Math. Gen.* **31** 5827
[7] Behrend R E, Pearce P A and Zuber J-B 1998 *J. Phys. A: Math. Gen.* **31** L763
[8] Behrend R E, Pearce P A, Petkova V B and Zuber J-B 1998 *Phys. Lett.* B **444** 163
[9] Zuber J-B 1986 *Phys. Lett.* B **176** 127
[10] Ruelle Ph and Verhoeven O 1998 *Nucl. Phys.* B **535** 650
[11] Ishibashi N 1989 *Mod. Phys. Lett.* A **4** 251
[12] Di Francesco P and Zuber J-B 1993 *J. Phys. A: Math. Gen.* **26** 1441
[13] Cardy J L and Lewellen D C 1991 *Phys. Lett.* B **259** 274
[14] Lewellen D C 1992 *Nucl. Phys.* B **372** 654
[15] Runkel I 1999 *Nucl. Phys.* B **549** 563
[16] Seneta E 1973 *Non-Negative Matrices* (London: George Allen and Unwin)
Minc H 1988 *Nonnegative Matrices* (New York: Wiley)
[17] Pearce P A 1999 Private communication
[18] Behrend R E *et al* in preparation